Indian Statistical Institute
M.Math II Year
Second Semester 2006-07
Mid Semester Examination
Algebraic Geometry

Time: 3 hrs Date:07-03-07 Total Marks: 50

Attempt all questions. All questions carry equal marks.

Assume k is an algebraically closed field **except** for question (1).

1. Let k be any field (not necessarily algebraically closed) and $\mathcal{M}$ a maximal ideal of $k[x_1,\ldots,x_n]$. Prove that $k[x_1,\ldots,x_n]/\mathcal{M}$ is a finite field extension of k. Let L be any extension field of k and let $V_{L/k}(\mathcal{M}) := \{P \in \mathbb{A}^n_L | f(P) = 0 \ \forall \ f \in \mathcal{M}\}$. Prove that $V_{L/k}(\mathcal{M})$ is a finite set (possibly empty).

2. Let $Y \subseteq \mathbb{A}^n_k$ be an affine variety. Identify $\mathbb{A}^n_k$ with $U_0 = \{x_0 \neq 0\} \subseteq \mathbb{P}^n_k$ via the isomorphism $\varphi_0 : \mathbb{A}^n_k \to U_0$, $\varphi(x_1,\ldots,x_n) = (1 : x_1 : \ldots : x_n)$. Define the projective closure, $\bar{Y}$, of Y to be the closure of Y in $\mathbb{P}^n_k$.
   (a) Show that $I(\bar{Y})$ is equal to the homogeneous ideal generated by $\beta(I(Y))$ in $k[x_0,\ldots,x_n]$, where β is the homogenization map w.r.t x_0, i.e., $\beta(f(x_1,\ldots,x_n)) = x_0^{\deg f} f(\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0})$.
   (b) Let $Y = \{(t,t^2,t^3) \in \mathbb{A}^3_k | t \in k\}$. Prove that $Y \subseteq \mathbb{A}^3_k$ is an affine variety. Find generators for $I(Y)$ and $I(\bar{Y})$. Is $\bar{Y}$ a variety? Describe $\bar{Y} - Y$.

3. Let $X = V(x - yz) \subseteq \mathbb{A}^3_k$. Consider the projection morphism $\varphi : X \to \mathbb{A}^2_k$, $\varphi(x,y,z) = (x,y)$, and show that there exists lines $L_1 \subseteq X$ and $L_2 \subseteq \mathbb{A}^2_k$ for which $X - L_1$ is isomorphic (via φ) to $\mathbb{A}^2_k - L_2$.

4. Does there exist a surjective morphism from $\mathbb{A}^1_k$ to $\mathbb{A}^1_k - \{0\}$? Does there exist a surjective morphism from $\mathbb{A}^1_k - \{0\}$ to $\mathbb{A}^1_k$?

5. Prove that the ring of regular functions on $\mathbb{A}^2_k - \{(0,0)\}$ is isomorphic to the polynomial ring over k in 2 variables.